B. Sc. (Part - III) (Semester - VI) (CBCS) Examination, June – 2022

MATHEMATICS

DSE – F10: Linear Algebra (Paper – IV)

Sub. Code: 81663

Question Bank

MULTIPLE CHOICE QUESTIONS

Q. Select the correct alternative for each of the following.

1. A zero vector is always ______.

    i) linearly dependent   ii) linearly independent

    iii) member of any basis   iv) none of these

2. The number of vectors in any basis of a vector space V is called ________ of V.

    i) rank   ii) nullity   iii) order   iv) dimension

3. If $T: U \to V$ is a linear transformation such that dim $U = 4$ and nullity $T = 2$ then rank of T is ______.

    i) 1   ii) 2   iii) 0   iv) 4

4. If $T: R^2 \to R^2$ and $S: R^2 \to R^3$ defined by $T(x, y) = (y, x)$ and $S(x, y) = (x + y, x - y, y)$ then $ST(x, y) =$ _______.

    i) $(y + x, y - x, x)$ ii) $(x - y, x + y, x)$

    iii) $(x - y, x + y, y)$ iv) $(y + 2x, y - x, x)$

5. If $u = (4, -3, -2, 1)$ then norm of u with respect to Euclidean inner product in $\mathbb{R}^4$ is _____.

    i) 30   ii) 26   iii) $\sqrt{30}$ iv) $\sqrt{26}$

6. If V is an inner product space and $u, v \in V$ such that u is orthogonal to v then _____.

    i) $\|u + v\|^2 = 0$ ii) $\|u + v\|^2 = \|u\|^2 + \|v\|^2$

    iii) $\|u + v\|^2 \leq \|u\|^2 - \|v\|^2$ iv) $\|u + v\|^2 \geq \|u\|^2 + \|v\|^2$

7. If $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ then the characteristic polynomial of A is ___________.

    i) $x^2 + 1$ ii) $x^2 + 2x + 1$ iii) $x^2 + x$ iv) $x^2 - 1$

8. If $\lambda = {}^1/_2$ is an eigen value of an invertible operator T then eigen value of T^{-1} is ______.

    i) -2 ii) ${}^1/_2$ iii) 2   +iv) $-{}^1/_2$

9. If W is a subspace of V then $L(W) =$ ________.

    i) W ii) V

    iii) $\{0\}$ iv) ϕ

10. If dim $V = n$ and $S = \{v_1, v_2, \dots, v_n\}$ spans V then S is ________ of V.

i) a subspace	ii) a basis

iii) a linearly dependent subset	iv) the smallest subspace

11. A linear transformation $T : V \to W$ is non singular if _________________.

i) T is not one- one	ii) T is not onto

iii) Ker $T = \{0\}$	iv) Range $T = \{0\}$

12. If T is a linear operator on R^2 defined by $T(x_1 , x_2) = (0 , 0)$ then rank of $T =$ ______.

i) 3	ii) 0

iii) 2	iv) 1

13. In an inner product space V, for any $u, v \in V$, $|(u, v)| \leq$ ___________.

i) $\|u\| + \|v\|$	ii) $\|u\|^2 . \|v\|^2$

iii) $\|u\| - \|v\|$	iv) $\|u\|.\|v\|$

14. A set $\{u_1, u_2, \dots, u_n\}$ of vectors in an Inner product space V is said to be orthogonal if ______.

i) $(u_i, u_j) = 0$ for all $i \neq j$	ii) $(u_i, u_j) \neq 0$ for all $i \neq j$

iii) $(u_i, u_i) = 0$ for all i	iv) $(u_i, u_i) = 1$ for all i

15. The eigen values of the matrix $\begin{bmatrix} -5 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ are ____________.

i) 1, 2, 3	ii) -5, 2, 0	iii) 3, 4, 6	iv) -5, 2, 1

16. If $T(1, 1) = (2, 2)$ then ______________ is an eigen value of T.

i) 0	ii) 1	iii) 2	iv) 3

17. Which of the following set is a linearly independent subset of $\mathbb{R}^3(\mathbb{R})$?

i) $\{(1, 0, 0), (2, 2, 0), (1, 1, 0)\}$	ii) $\{(1, 0, 0), (1, 1, 1), (0, 0, 0)\}$

iii) $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$	iv) $\{(1, 0, 0), (2, 1, 0), (1, 1, 0)\}$

18. Let $\{u, v, w\}$ be a linearly independent set in a vector space. Then which of the following is correct?

i) u is a linear combination of v and w.

ii) $\{u, v, u + v\}$ is linearly independent.

iii) $au + bv + cw = 0$ for some nonzero scalars a, b and c.

iv) $\{u, u + v, u + v + w\}$ is linearly independent.

19. Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear transformation and $\{e_1 = (1,0), e_2 = (0,1)\}$ be the standard basis of $\mathbb{R}^2$. If $T(e_1) = (2,1)$ and $T(e_2) = (1,3)$ then $T(x,y) =$ ________

i) $(2x + y,\ x)$ ii) $(2x\ ,\ 3y)$

iii) $(2x + y,\ x + 3y)$ iv) $(x + y,\ x - y)$

20. Let T be a linear operator on $\mathbb{R}^3$, defined by

$$T(x,y,z) = (x + z,\ \ x + y + 2z,\ \ 2x + y + 3z)$$

Then ________

i) $(3,3,-3) \in \operatorname{Ker} T$ ii) $(1,2,3) \in \operatorname{Ker} T$

iii) Ker T is the empty set iv) $\operatorname{Ker} T = \{0\}$

21. Let V be the inner product space of real polynomials of degree at most 2 with respect to the inner product defined by

$$\langle f,\ \ g\rangle = \int_0^1 f(x)\,.\,g(x)\,dx$$

If $f(x) = 2x$ and $g(x) = x^2$, then $\langle f, g\rangle =$ ________

i) 1 ii) -1 iii) 2 iv) $\frac{1}{2}$

22. If $\{w_1, w_2, \ldots \ldots, w_n\}$ is an orthonormal set in an inner product space V, then

$$\sum_{i=1}^{n} |\langle w_i\ , v\rangle|^2 \leq \|v\|^2 \text{ for all } v \in V$$

This property is known as ________

i) Sylvester's law ii) Cauchy – Schwarz inequality

iii) Triangle inequality iv) Bessel's inequality

23. If one Eigen value of the matrix $\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$ is 5 then the second is ________

i) 2 ii) – 2 iii) – 1 iv) – 5

24 Two matrices A, B are said to be similar matrices if there exists a non-singular matrix P such that ________

i) $B = P^{-1}AP$ ii) $B = P^{-1}PA$ iii) $A = B$ iv) $AB = PA$

25. Let $S = \{(-1,0,1), (2,1,4)\}$. The value of k for which the vector $(3k + 2, 3, 10)$ belongs to the linear span of S is ………

i) 2 ii) – 2 iii) 8 iv) 3

26. Which of the following is incorrect?

i) A basis of a vector space is a maximal linearly independent set.

ii) A minimal generating subset of a vector space V is a basis for V.

iii) Any two bases of a F. D. V. S. have same number of vectors.

iv) If $\dim V = n$, then any $n+1$ vectors in V are linearly independent.

27. Let $T: V \longrightarrow W$ be a linear transformation and dim Range T = 3 and dim $V = 8$. Then Nullity of $T =$ ________

i) 5 ii) 11 iii) 24 iv) 3

28. If $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + y,\ x)$ is an invertible linear transformation, then $T^{-1}(a,\ b) =$ _____

i) $(a,\ a-b)$ ii) $(a,\ a+b)$ iii) $(b,\ a+b)$ iv) $(b,\ a-b)$

29. If V is an Inner product space and $x, y \in V$ then $\|x+y\|^2 + \|x-y\|^2 =$ ______________.

i) $2(\|x\|^2 - \|y\|^2)$ ii) $\|x\|^2 + \|y\|^2$ iii) $2(\|x\|^2 + \|y\|^2)$ iv) $\|x\|^2 - \|y\|^2$

30. The norm of vector $u = (1, -2, 5)$ with respect to Euclidean inner product is __________

i) $\sqrt{30}$ ii) $\sqrt{22}$ iii) $2\sqrt{5}$ iv) $6\sqrt{5}$

31. Let c be an eigen value of a linear operator T on V. Then the set $\{v \in V \mid T(v) = cv\}$ is called ______ of T.

i) eigen space ii) null space

iii) range iv) kernel

32. The characteristic polynomial of the matrix $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$ is ________

i) $x^2 - 2x + 3$ ii) $x^2 + 3x - 10$

iii) $x^2 - 3x$ iv) $x^2 - 3x - 10$

33. If $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$,where $v_1, v_2, \ldots, v_n$ are linearly independent vectors in a vector space $V(F)$, then _________.

i) $\alpha_i = 0$ for all $i = 1,2, \ldots, n$ ii) $\alpha_i \neq 0$ for all $i = 1,2, \ldots, n$

iii) $\alpha_i = 0$ for exactly one i iv) $\alpha_i \neq 0$ for at least one i

34. The set $S = \{(2,\ 4),\ (-1,\ 3),\ (6, -7)\}$ of vectors in R^2 is _________ .

i) a linearly independent subset ii) a basis of R^3

iii) a linearly dependent subset iv) an orthogonal set

35. The identity transformation $I: V \to V$ defined by $I(v) = v, \ \forall v \in V$ then Kernel of $I =$ ________.

i) V ii) $\{0\}$

iii) the empty set iv) None of these

36. If $T: V \to W$ and $S: W \to U$ are two linear transformations such that ST is onto then _______.

i) S is onto ii) T is one – one

iii) S is one – one iv) T is onto

37. If $u = (-1, 1, 2)$ and $v = (2, 1, 0)$ then $||u + v|| =$ ___________.

i) $\sqrt{13}$ ii) 13

iii) $\sqrt{3}$ iv) 3

38. For all u, v in an inner product space V, the inequality $||u + v|| \leq ||u|| + ||v||$ is called __________ inequality.

i) Cauchy - Schwarz ii) Minkowski

iii) Cauchy iv) Triangle

39. The characteristic polynomial of the matrix $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ is ____________.

i) x^2 ii) $x^2 + 1$

iii) $x^2 - 1$ iv) $(x - 1)^2$

40. The constant term of the characteristic polynomial of a square matrix A is ________.

i) 1 ii) 0

iii) trace of A iv) $(-1)^n \det A$

---*@*---

8 Marks Questions

1. Define a subspace of a vector space. Prove that a non-empty subset W of a vector space $V(F)$ is a subspace of V if and only if $\alpha x + \beta y \in W$ for $\alpha, \beta \in F$ and $x, y \in W$.
2. Let V and W be two vector spaces over F. Let $\{v_1, v_2, \dots, v_n\}$ be a basis of V and $w_1, w_2, \dots, w_n$ be any vectors in W. Then prove that there exists a unique linear trasnsformation $T: V \to W$ such that $T(v_i) = w_i$, $i = 1, 2, \dots, n$.
3. Define an inner product space. If V is an inner product space then prove that
   (i) $\|u + v\| \leq \|u\| + \|v\|$ and
   (ii) $\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$ for all $u, v \in V$.
4. Define a subspace of a vector space. Prove that a necessary and sufficient condition for a non-empty subset W of a vector space $V(F)$ to be a subspace of V is that W is closed under addition and scalar multiplication.
5. State and prove rank – nullity theorem.
6. Let V be a non-trivial inner product space of dimension n. Prove that V has an orthonormal basis.
7. Let V be a vector space and S be a non-empty subset of V. Prove that the linear span $L(S)$ is the smallest subspace of V containing S.
8. Let V be an inner product space. Then prove that $|(u,\ v)| \leq \|u\|.\|v\|$ for all $u, v \in V$.
9. Let $T: V \to U$ be a linear transformation then prove that
$$\frac{V}{Ker\ T} \cong Range\ T = T(V)$$
10. If V is a F.D.V.S. and $\{v_1,\ v_2,\ v_3,\ \dots\ , v_r\}$ is a Linearly independent subset of V, then show that it can be extended to form a basis of V.
11. If A and B are two subspaces of a vector space $V(F)$, then
$$\frac{A + B}{A} \cong \frac{B}{A \cap B}$$
12. Find eigen values and eigen vectors of the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$.
13. If $\{w_1, w_2, \dots \dots, w_n\}$ is an orthonormal set in an inner product space V, then prove that
$$\sum_{i=1}^{n} |\langle w_i\ , v\rangle|^2 \leq \|v\|^2 \ \text{ for all } v \in V$$

14. Prove that a Linear transformation $T : V \rightarrow W$ is non-singular if and only if T carries each Linearly independent subset of V onto a Linearly independent subset of W.
15. Let S be a finite subset of a vector space V such that $V = L(S)$ then prove that there exists a subset of S which forms a basis of V.

---*@*---

<u>4 Marks Questions</u>

1. Determine whether the vectors $(1,0,1),(1,1,0),(-1,0,1)$ are linearly dependent or linearly independent.
2. Define the kernel of a homomorphism. Prove that the kernel of a homomorphism $T: V \rightarrow U$ is a subspace of V.
3. Show that the linear operator T on R^3 defined by $T(x,y,z) = (x+y,\ y+z,\ x+y+z)$ is invertible and find its inverse.
4. If S is an orthogonal set of non-zero vectors in an inner product space V then prove that S is a linearly independent set.
5. Obtain an orthonormal basis with respect to the standard inner product for R^3 generated by $(1,0,0)$, $(1,1,1)$ and $(1,2,3)$.
6. Find eigen values of the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$.
7. If $T: V \rightarrow U$ is a linear transformation then prove that Ker $T = \{0\}$ if and only if T is one – one.
8. Show that intersection of two subspaces of a vector space V is a subspace of V.
9. Find the rank and nullity of the linear transformation $T: R^2 \rightarrow R^3$ defined by $T(x,y) = (x,\ x+y,\ y)$.
10. Let V be an inner product space. Then prove that $\|x+y\| \leq \|x\| + \|y\|$, for all $x, y \in V$.
11. Show that the vectors $(1,-2,\ 3)$, $(5,6,-1)$ and $(3,2,1)$ are linearly dependent in R^3.
12. If $c \neq 0$ is an eigen value of an invertible operator T then prove that c^{-1} is an eigen value of T^{-1}.
13. If $T: R^3 \rightarrow R^3$ is defined as $T(x,y,z) = (x,\ x+y,\ x+y+z)$, then show that T is a linear transformation.
14. Show that the vectors $(1,0,1),(0,1,1),(1,1,1)$ are linearly independent in R^3.

15. Define the Range of a linear transformation. Prove that the Range of a linear transformation $T: V \to U$ is a subspace of U.
16. Let T be a linear operator on a finite dimensional vector space V over F. Then prove that $c \in F$ is an eigen value of T if and only if $T - cI$ is singular.
17. Obtain an orthonormal basis with respect to the standard inner product for the subspace of R^4 generated by $(1,\ 0,\ 2,\ 0),\ (1,\ 2,\ 3,\ 1)$.
18. Find eigen values of the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$.
19. Show that the sum of two subspaces of a vector space $V(F)$ is a subspace of V.
20. If $T: V \to U$ is a homomorphism, then show that

    (i) $T(0) = 0$ (ii) $T(-x) = -T(x)$
21. Determine whether or not $W = \{(a, b, c) \in \mathbb{R}^3 : b = a^2\}$ is a subspace of $\mathbb{R}^3$.
22. If $\dim V = n$, then show that any $n + 1$ vectors in V are linearly dependent.
23. Let V be an inner product space. Then prove that for all $x, y \in V$,
$$\|x + y\|^2 + \|x - y\|^2 = 2\,(\|x\|^2 + \|y\|^2)$$
24. Let T be a linear operator on a vector space V. Define of Eigen space of T associated with Eigen value c. Show that the Eigen space is a subspace of V.
25. If S_1 and S_2 are subsets of a vector space V, then show that $L(S_1 \cup S_2) = L(S_1) + L(S_2)$.
26. Determine whether or not $F: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $F(x, y, z) = (|x|\,,\ y + z)$ is a linear transformation.
27. Let $T : V \to W$ and $S : W \to U$ be two linear transformations. If ST is one – one then prove that T is one-one.
28. If $T : V \to V$ be a linear transformation, then prove that the following statements are equivalent:

    (i) $Range\, T \cap Ker\, T = \{0\}$

    (ii) If $T\big(T(v)\big) = 0$ then $T(v) = 0,\ v \in V$.
29. Let $u = (-2,\ -1,\ 4,\ 5)\,, v = (3,\ 1, -5,\ 7)\,, w = (-6,\ 2,\ 1,\ 1)$ in Euclidean inner product space $\mathbb{R}^4$. Find (a) $\|4u - 2v + w\|$ and (b) $\big\|\,\|u - v\|\, w\,\big\|$.
30. Let P_2 be the inner product space of polynomials of degree at most 2 with respect to the inner product defined as:
$$(p,\ q) = \int_{-1}^{1} p(x)\, q(x)\, \mathrm{dx} \quad \forall\, p, q \in P_2$$
    Show that $p = x$ and $q = x^2$ are orthogonal in P_2. Find $\|p + q\|^2$.